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Exact results for one-dimensional totally asymmetric diffusion models

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Abstract. Several types of totally asymmetric diffusion models with and without exclusion are considered. For some models, conditional probabilities of finding N particles on lattice sites $x_1, \dots, x_N$ at time t with initial occupation $y_1, \dots, y_N$ at time $t = 0$ are expressed in a determinant form. On the other hand, the q-boson totally asymmetric diffusion model is introduced which interpolates the free boson model and the model with exclusion-like interaction. The effects of the interaction are compared for the case of two-particle diffusion.

1. Introduction

Recently, one-dimensional stochastic models have been intensively studied [1, 2]. We may list reasons of the motivations. First, the models are useful in the study of non-equilibrium statistical physics. They are defined by rather simple rules but are sufficiently complex so as to exhibit rich non-equilibrium behaviours. Particularly in low dimensions, the effects of fluctuations are so strong that the system cannot be described by a simple mean-field theory and more elaborated treatments are needed. Second, applicability of the models is not restricted to physics. They are also useful in many fields of sciences such as chemistry, biology and social science. There are many interesting stationary and time-dependent properties of 'open' systems to be explained. Third, we can relate the seemingly different subjects in statistical physics. While the models are suitable for numerical studies, some models can be solved exactly by analytical methods such as the Bethe ansatz and the free fermion theory. The models provide many good examples to investigate how far applicability of such techniques can be extended without losing real physical significance.

In this paper, we deal with the one-dimensional asymmetric diffusion models. The models are physically relevant to describe the gas flows in a narrow tube driven by some external field. The asymmetric simple exclusion process (ASEP) is the most famous example of such models [3, 4]. The model has applications to a variety of interesting phenomena from interface growth to traffic flow [2, 5]. It is a model for particles on a lattice, where each particle hops to the right (left) nearest-neighbouring site with the probability $D_R \mathrm{d}t$ ($D_L \mathrm{d}t$) in every infinitesimal time interval $\mathrm{d}t$. In addition, each site of the lattice can contain only one particle: each site is either occupied or empty. This constraint is interpreted as a hard-core exclusion among particles and is regarded as the origin of the interesting behaviours of the model. In particular, the ASEP is considered to belong to the Kardar–Parisi–Zhang (KPZ) universality class from the analysis of the Bethe ansatz equations [6–9]. Although many exact results are reported on this model, the time-dependent properties of the model have not been well understood.

In [10], the N-particle Green's function for the ASEP on an infinite lattice is shown to be written in a determinant form. This expression is valid only for the totally asymmetric case ($D_R = 1, D_L = 0$), which we refer to as the TASEP. Very recently, the determinant representations for Green's function have been obtained for the drop-push model [12] and a derivative nonlinear Schrödinger-type model [13]. A main purpose of this paper is to investigate in detail the determinant form Green's function. It will be shown that such formula is valid for a wide class of totally asymmetric diffusion models.

One of the models with the determinant formula is a model which is written in terms of the so-called phase operators. We call the model the phase model hereafter. Whereas particles also hop in one direction in this model, the interaction of the particles is slightly different from the TASEP. The model was introduced in [14] and was shown to be integrable with the same R-matrix as that for the TASEP. Hence, although particles do not have the exclusion property in this model, the model is expected to be in the same universality class as the TASEP. In this paper, we show that, at least for the two-particle problem, these two models show similar behaviours in the asymptotic regime.

In addition, we introduce the q-boson totally asymmetric diffusion model, which is slightly different from the model in [15–18]. We show the integrability of the model and study the two-particle diffusion. This model is interesting since it contains the free boson model and the phase model as special limits. Hence, the q-boson model to be considered can be an appropriate candidate to study the crossover behaviours from the single-particle asymmetric random walk to the KPZ universality class. Such crossover behaviours may be observed experimentally, for instance, by varying the radius of the tube in which a gas flows. When the radius of the tube is large, interactions among particles are scarce and the dynamics of each particle is well described by the single-particle asymmetric random walk. On the other hand, for a very narrow tube, the system is expected to be in the KPZ universality class since exclusion interactions have significant effects on the properties of the system. In the q-boson model, we can consider that the radius of the tube is efficiently taken into account through the value of the parameter q.

This paper is organized as follows. Several types of totally asymmetric diffusion models are defined in section 2. In section 3, it is shown that we can obtain the determinant form Green's function for a wide class of totally asymmetric diffusion models. In section 4, the eigenvalue problem for the TASEP-type model is solved by the Bethe ansatz method. Using the obtained expression, we study the two-particle diffusion problem. As a model which connects the phase model and the free boson model, we introduce the q-boson totally asymmetric diffusion model in section 5. The integrability is proved and the two-particle diffusion problem is investigated. The final section is devoted to concluding remarks.

2. Definition of the models

Consider a system of particles where they move and interact stochastically on a lattice in one dimension. Time evolution for the system can be described by the master equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = -H|P(t)\rangle. \tag{2.1}$$

Here H is a time evolution operator specified by the rules for the process and the state $|P(t)\rangle$ represents the probability distribution of the system. Although the operator H is in general non-Hermitian, it will be called a Hamiltonian hereafter. We may consider (2.1) as an (imaginary-time) Schrödinger equation.

When we consider a system of exclusive particles, i.e. a case where at most one particle is allowed to be on each lattice site, it is useful to express the Hamiltonian in the spin-$\frac{1}{2}$ variables [19]. We identify the 'unoccupied' state with 'spin-up' state and the 'occupied' state with the 'spin-down' state. The ASEP is a model of such exclusive particles, each of which diffuses to the right nearest-neighbouring site with rate D_R and to the left nearest-neighbouring site with rate D_L. In this paper, we restrict our attention mainly to the TASEP, where particles hop only in one direction ($D_R = 1, D_L = 0$). The Hamiltonian for the TASEP in the spin-$\frac{1}{2}$ variables is given by

$$H_{\mathrm{TASEP}} = -\sum_j [s_j^+ s_{j+1}^- - n_j(1 - n_{j+1})] \tag{2.2}$$

where $s_j^{\pm}, n_j$ are defined through the Pauli's matrices $\sigma_j^{x,y,z}$ by $s_j^{\pm} = \frac{1}{2}(\sigma_j^x \pm \mathrm{i}\sigma_j^y), n_j = \frac{1}{2}(1 - \sigma_j^z)$. As a natural generalization of the TASEP, we define 'm-TASEP' for which the Hamiltonian is given by

$$H_{m\text{-TASEP}} = -\sum_j [s_j^+ s_{j+1}^- - n_j(1 - n_{j+1})](1 - n_{j+2}) \ldots (1 - n_{j+m}). \tag{2.3}$$

The original TASEP corresponds to $m = 1$. As can be seen from the Hamiltonian, each particle can hop to the right nearest-neighbouring site only when the distance to the right nearest particle is greater than or equal to m. We notice that the integer m is regarded as the length (size) of each particle.

On the other hand, when we consider a system of non-interacting particles without exclusion, the Hamiltonian is expressed in terms of the boson operators[20, 21]. The Hamiltonian is provided by

$$H_{\mathrm{boson}} = -\sum_j (b_{j+1}^\dagger - b_j^\dagger) b_j \tag{2.4}$$

where creation operators $b_j^\dagger$ and annihilation operators b_j obey the bosonic commutation relations $[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0$, $[b_j, b_k^\dagger] = \delta_{jk}$. Since the Hamiltonian (2.4) is quadratic in terms of the boson operators, we call this model the free boson model.

In this paper, we also study another model of interacting particles without exclusion. The model is defined in terms of the q-boson operators $B_j^\dagger$ and B_j. With the number operator N_j, these operators form the following algebra,

$$\begin{aligned} &[N_j, B_k^\dagger] = B_j^\dagger \delta_{jk} \qquad [N_j, B_k] = -B_j \delta_{jk} \\ &[B_j, B_k^\dagger] = q^{-2N_j} \delta_{jk}. \end{aligned} \tag{2.5}$$

In terms of these operators, a totally asymmetric diffusion model can be defined by the Hamiltonian,

$$H_{q\text{-boson}} = -\sum_j (B_{j+1}^\dagger - B_j^\dagger) B_j. \tag{2.6}$$

In [15–18], a slightly different q-boson Hamiltonian was considered. Their Hamiltonian is, however, not stochastic except when $q \to 1$. Our Hamiltonian (2.6) is stochastic for any $q > 0$. When we take the limit $q \to 1$, the q-bosons become ordinary bosons and the q-boson Hamiltonian (2.6) reduces to the free boson Hamiltonian (2.4).

We can take another limit $q \to \infty$. In this limit, it is known that the q-boson operators become the so-called phase operators [14, 15]. They are defined by the following commutation relations,

$$
\begin{aligned}
&[N_j, \phi_k^\dagger] = \phi_j^\dagger \delta_{jk} \qquad [N_j, \phi_k] = -\phi_j \delta_{jk} \\
&[\phi_j, \phi_k^\dagger] = \pi_j \delta_{jk}
\end{aligned}
\tag{2.7}
$$

where π_j is the vacuum projector $\pi_j = |0\rangle_j \langle 0|_j$. The Hamiltonian is given by

$$
\begin{aligned}
H_{\text{phase}} &= -\sum_j (\phi_{j+1}^\dagger - \phi_j^\dagger)\phi_j \\
&= -\sum_j (\phi_{j+1}^\dagger \phi_j + \pi_j) + \text{constant}
\end{aligned}
\tag{2.8}
$$

where we used $\phi_j^\dagger \phi_j = 1 - \pi_j$. As recently shown by Bogoliubov *et al* [14, 15], the phase model (2.8) is integrable with the same R-matrix as that for the TASEP. Hence the model is expected to belong to the KPZ universality class.

3. Determinant form solution for the master equation

For the TASEP and some related models, it has been shown that the Green's function for N particles can be expressed in a determinant form [10, 12, 13]. In this section, we show that the similar determinant representation for the Green's function is realized for a wide class of totally asymmetric diffusion models. Let us denote the Green's function for N particles as $P(x_1, \ldots, x_N; t|y_1, \ldots, y_N; 0)$. It is defined as the solution $P(x_1, \ldots, x_N; t)$ of the master equation $\partial P/\partial t = -HP$ with the initial condition,

$$
P(x_1, \ldots, x_N; t = 0) = \delta_{x_1 y_1} \ldots \delta_{x_N y_N}. \tag{3.1}
$$

In statistical mechanics, it is the conditional probabilities of finding N particles on lattice sites $x_1, \ldots, x_N$ at time t with initial occupation $y_1, \ldots, y_N$ at time $t = 0$.

3.1. m-TASEP

In this section, the m-TASEP (2.3) is considered. First we write down the master equation for the process. Before discussing the general N-particle case, we proceed with the $N = 1$ and $N = 2$ cases. For $N = 1$, the equation is given by

$$
\frac{\partial}{\partial t} P(x; t) = P(x - 1; t) - P(x; t). \tag{3.2}
$$

This is nothing but the Poisson process and the Green's function is given by

$$
P(x; t|y; 0) = \frac{t^{x-y}}{(x - y)!} \mathrm{e}^{-t}. \tag{3.3}
$$

Here the factorial is defined by the Γ-function, i.e. $x! = \Gamma(x + 1)$. Since $\Gamma(x)$ has poles at non-positive integers, expression (3.3) vanishes for the region $x < y$. This corresponds to the fact that the particle never hops to the left.

Next we consider the $N = 2$ case. Notice that the integer m restricts an allowed region (physical region) of the coordinates x_1, x_2. The physical region for the m-TASEP is $x_2 - x_1 \geqslant m$. When $x_2 - x_1 > m$, the master equation reads

$$
\frac{\partial}{\partial t} P(x_1, x_2; t) = P(x_1 - 1, x_2; t) + P(x_1, x_2 - 1; t) - 2P(x_1, x_2; t). \tag{3.4}
$$

At the boundary of the physical region (i.e. when $x_2 - x_1 = m$), the master equation depends on the integer m. For the m-TASEP, it reads

$$\frac{\partial}{\partial t} P(x_1, x_1 + m; t) = P(x_1 - 1, x_1 + m; t) - P(x_1, x_1 + m; t). \tag{3.5}$$

The master equation at the boundary of the physical region can be replaced by the boundary condition for the probability distribution $P(x_1, x_2; t)$. In other words, equation (3.5) is equivalent to putting the boundary condition,

$$P(x_1, x_1 + m - 1; t) = P(x_1, x_1 + m; t). \tag{3.6}$$

Similarly, the master equation for general N-particle case is given by

$$\begin{aligned}\frac{\partial}{\partial t} P(x_1, \ldots, x_N; t) &= P(x_1 - 1, \ldots, x_N; t) + \cdots + P(x_1, \ldots, x_N - 1; t) \\ &\quad - N P(x_1, \ldots, x_N; t)\end{aligned} \tag{3.7}$$

with the boundary condition,

$$P(\ldots, x_j, x_j + m - 1, \ldots; t) = P(\ldots, x_j, x_j + m, \ldots; t) \qquad (j = 1, \ldots, N - 1). \tag{3.8}$$

Until now, the value of the integer m has been assumed to be greater than or equal to 1 implicitly. However, it turns out that the boundary condition for the phase model (2.8) is nothing less than equation (3.6) for $m = 0$. Hence we can deal with the phase model within the same formulation assuming that the value of the integer m is $m \geqslant 0$. As a comment, we notice that we can also consider the $m < 0$ case if we distinguish the N particles.

The N-particle Green's function for the m-TASEP can be written in a determinant form:

$$P(x_1, \ldots, x_N; t | y_1, \ldots, y_N; 0) = \det F \tag{3.9}$$

where the matrix elements of the matrix F are assumed to be of the form,

$$F_{jk} = F_{j-k}(x_k - y_j; t). \tag{3.10}$$

In a visual fashion, relation (3.9) with (3.10) looks like

$$\begin{aligned} &P(x_1, \ldots, x_N; t | y_1, \ldots, y_N; 0) \\ &\quad = \begin{vmatrix} F_0(x_1 - y_1; t) & F_{-1}(x_2 - y_1; t) & \cdots & F_{-N+1}(x_N - y_1; t) \\ F_1(x_1 - y_2; t) & F_0(x_2 - y_2; t) & \cdots & F_{-N+2}(x_N - y_2; t) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N-1}(x_1 - y_N; t) & F_{N-2}(x_2 - y_N; t) & \cdots & F_0(x_N - y_N; t) \end{vmatrix}. \end{aligned} \tag{3.11}$$

There is an arbitrariness for the function $F_n(x; t)$. If we define $\hat{F}_n(x; t) = a^n F_n(x; t)$ for a non-zero a, it gives the same value for the determinant: $\det F = \det \hat{F}$. The parameter a will be chosen following [10].

Let $F_n^{(m)}(x; t)$ denote $F_n(x; t)$ for the m-TASEP. For the TASEP ($m = 1$), the solution is known to be

$$F_n^{(1)}(x; t) = \mathrm{e}^{-t} \sum_{l=0}^{\infty} \begin{bmatrix} l - n - 1 \\ l \end{bmatrix} \frac{t^{x+l}}{(x + l)!}. \tag{3.12}$$

We list the properties of the function $F_n^{(1)}(x; t)$. It has an integral representation,

$$F_n^{(1)}(x; t) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}p\, (1 - \mathrm{e}^{-\mathrm{i}(p+\mathrm{i}0)})^n \mathrm{e}^{\mathrm{i}px - \epsilon_p t} \tag{3.13}$$

where $\epsilon_p = 1 - \mathrm{e}^{-\mathrm{i}p}$. The shift $p \to p + \mathrm{i}0$ is chosen in order to satisfy the initial condition (3.1). It satisfies the master equation for the $N = 1$ case (cf (3.2)),

$$\partial_t F_n^{(1)}(x; t) = F_n^{(1)}(x - 1; t) - F_n^{(1)}(x; t). \tag{3.14}$$

For $n \geqslant 0$, it reduces to a finite sum,

$$F_n^{(1)}(x;t) = \mathrm{e}^{-t} \sum_{l=0}^{n} (-)^l \begin{bmatrix} n \\ l \end{bmatrix} \frac{t^{x+l}}{(x+l)!}. \tag{3.15}$$

In addition, we can calculate the value for the limit $t \to +0$. For $n \geqslant 0$, we find

$$F_n^{(1)}(x;0) = \sum_{l=0}^{n} (-)^l \begin{bmatrix} n \\ l \end{bmatrix} \delta_{x+l,0} \tag{3.16}$$

whereas for $n < 0$, we have

$$F_n^{(1)}(x;0) = \alpha(x) \begin{bmatrix} -n-x-1 \\ -x \end{bmatrix} \tag{3.17}$$

where $\alpha(x) = 0 (x > 0)$, $1 (x \leqslant 0)$.

The $F_n^{(m)}(x;t)$ for the m-TASEP is simply given by

$$F_n^{(m)}(x;t) = F_n^{(1)}(x + n(m-1);t). \tag{3.18}$$

The proof is almost the same as for the $m = 1$ case and the main points are summarized as follows. First the master equation (3.7) is satisfied since $F_n^{(m)}(x;t)$ satisfies the master equation for $N = 1$. Second the boundary condition (3.8) is fulfilled since the $F_n^{(m)}(x;t)$ satisfies

$$F_n^{(m)}(x+m-1;t) - F_n^{(m)}(x+m;t) = F_{n+1}^{(m)}(x;t). \tag{3.19}$$

Finally, it can be shown by using (3.16) and (3.17) that, when $t \to +0$, the determinant satisfies the initial condition (3.1) in the physical region.

3.2. *Drop-push-type model*

Recently it has been shown that it is possible to derive the determinant formula of the Green's function for the drop-push model [11, 12]. The Hamiltonian of the model is given by

$$H_{\text{drop-push}} = -\sum_j \sum_{k=1}^{\infty} [s_j^+ n_{j+1} \ldots n_{j+k-1} s_{j+k}^- - n_j n_{j+1} \ldots n_{j+k-1}(1 - n_{j+k})]. \tag{3.20}$$

In this model, even if the right nearest-neighbouring site is occupied, a particle hops to the site, pushing the right neighbouring particles to the next sites. The processes such as $10 \to 01$, $110 \to 011$, $1110 \to 0111, \ldots$ occur with equal rate, where '0' and '1' indicates the empty site and the occupied site. In terms of the boundary condition for the probability distribution $P(x_1, \ldots, x_N; t)$, the model corresponds to the condition,

$$P(\ldots, x_j, x_j+1, \ldots; t) = P(\ldots, x_j+1, x_j+1, \ldots; t) \qquad (j = 1, \ldots, N-1). \tag{3.21}$$

We can generalize the model by considering the boundary condition,

$$P(\ldots, x_j, x_j+m, \ldots; t) = P(\ldots, x_j+1, x_j+m, \ldots; t) \qquad (j = 1, \ldots, N-1). \tag{3.22}$$

We call this model the m-drop-push model in the following. The original drop-push model corresponds to the choice $m = 1$. For N-particle Green's function of the m-drop-push model, a similar situation occurs as for the m-TASEP. If we denote $F_n(x;t)$ for the m-drop-push model by $\tilde{F}_n^{(m)}(x;t)$, it is simply given by

$$\tilde{F}_n^{(m)}(x;t) = \tilde{F}_n^{(1)}(x + n(m-1);t). \tag{3.23}$$

Here $\tilde{F}_n^{(1)}(x;t)$ is $F_n(x;t)$ for the drop-push model,

$$\tilde{F}_n^{(1)}(x;t) = (-)^n \mathrm{e}^{-t} \sum_{l=0}^{\infty} \begin{bmatrix} l+n-1 \\ l \end{bmatrix} \frac{t^{x-l}}{(x-l)!} \tag{3.24}$$

and is given in an integral representation as

$$\tilde{F}_n^{(1)}(x;t) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}p\, (\mathrm{e}^{-\mathrm{i}(p-\mathrm{i}0)} - 1)^{-n} \mathrm{e}^{\mathrm{i}px - \epsilon_p t}. \tag{3.25}$$

In this expression, we have put $p \to p - \mathrm{i}0$ in order to satisfy the initial condition. Then we can calculate the limit $t \to +0$. For $n \leqslant 0$, we have

$$\tilde{F}_n^{(1)}(x;0) = \sum_{l=0}^{-n} (-)^l \begin{bmatrix} -n \\ l \end{bmatrix} \delta_{x+n+l,0} \tag{3.26}$$

and for $n > 0$, we find

$$\tilde{F}_n^{(1)}(x;0) = (-)^n \theta(x) \begin{bmatrix} n+x-1 \\ x \end{bmatrix} \tag{3.27}$$

where $\theta(x) = 1 (x \geqslant 0)$, $0 (x < 0)$. The proof that the determinant (3.9) with the choice (3.23) gives the N-particle Green's function is almost the same as for the m-TASEP case and is therefore omitted.

We notice that the $\tilde{F}_n^{(m)}(x;t)$ for $n \leqslant 0$ is equivalent to $F_{-n}^{(1-m)}(x;t)$:

$$\tilde{F}_n^{(m)}(x;t) = F_{-n}^{(1-m)}(x;t) \qquad (n \leqslant 0). \tag{3.28}$$

It is extremely interesting to observe that the same function appears in the Green's function for different models, i.e. for the m-drop-push model and the $(1-m)$-TASEP.

3.3. *Continuum model*

In [13], it was shown that a derivative nonlinear Schrödinger (DNLS) type model can be regarded as a continuum limit of the ASEP. The Hamiltonian of the model is given by

$$H_{\mathrm{DNLS}} = -\int \mathrm{d}x\, \psi^\dagger(x) \partial_x^2 \psi(x) + 2\alpha \int \mathrm{d}x\, \psi^\dagger(x) \partial_x \psi^\dagger(x) \psi(x) \psi(x) \tag{3.29}$$

where α is a real parameter. The operators ψ and $\psi^\dagger$ obey the commutation relations for bosons; $[\psi(x), \psi(y)] = [\psi^\dagger(x), \psi^\dagger(y)] = 0, [\psi(x), \psi^\dagger(y)] = \delta(x-y)$ with $\delta(x)$ being the δ-function. The master equation for this model reads

$$\frac{\partial}{\partial t} P(x_1, \ldots, x_N; t) = \sum_{j=1}^{N} \partial_j^2 P(x_1, \ldots, x_N; t)$$
$$+ 2\alpha \sum_{j<k} \delta(x_j - x_k)(\partial_j + \partial_k) P(x_1, \ldots, x_N; t). \tag{3.30}$$

In view of the initial condition, the Kronecker's delta in (3.1) is replaced by the Dirac's delta function,

$$P(x_1, \ldots, x_N; 0) = \delta(x_1 - y_1) \ldots \delta(x_N - y_N). \tag{3.31}$$

It has been shown that for $\alpha = 1$ we can represent the N-particle Green's function in a determinant form [13].

It turns out that the determinant formula is also valid for the $\alpha = -1$ case. The details will not be displayed since the discussion proceeds almost in the same way as for the $\alpha = 1$

case. As will become clear, this choice may be regarded as a continuum limit of the drop-push model. When $\alpha = -1$, the second sum in the right-hand side of (3.30) is equivalent to putting the following boundary conditions,

$$\partial_j P(\ldots, x_j, x_{j+1}, \ldots)|_{x_j = x_{j+1}} = 0 \qquad (j = 1, \ldots, N-1). \tag{3.32}$$

If we denote the function $F_n(x;t)$ for $\alpha = -1$ case as $\tilde{F}_n(x;t)$, the integral representation of it is given by

$$\tilde{F}_n(x;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p\,(p - \mathrm{i}0)^{-n} \mathrm{e}^{-p^2 t + \mathrm{i}px}. \tag{3.33}$$

We notice the similarity between (3.25) and (3.33). We can obtain more explicit expressions. For $n \leqslant 0$, we have

$$\tilde{F}_n(x;t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{4t}} (2t)^{(n-1)/2} (\mathrm{i})^{-n} H_{-n}\left(\frac{x}{\sqrt{2t}}\right) \tag{3.34}$$

where $H_n(x)$ are the Hermite polynomials. These functions appear in the Green's function for the $\alpha = 1$ case as well [13]. The situation is analogous to relation (3.28). For $n > 0$ we can obtain the explicit expression for $\tilde{F}_n(x;t)$ by using the recursion relation,

$$\tilde{F}_n(x;t) = \frac{\mathrm{i}x}{n-1}\tilde{F}_{n-1}(x;t) - \frac{2t}{n-1}\tilde{F}_{n-2}(x;t). \tag{3.35}$$

For instance, the first two functions are given explicitly as

$$\begin{aligned} \tilde{F}_0(x;t) &= \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-x^2/4t} \\ \tilde{F}_1(x;t) &= \frac{\mathrm{i}}{\sqrt{4\pi t}} \int_{-\infty}^{x} \mathrm{d}y\, \mathrm{e}^{-y^2/4t}. \end{aligned} \tag{3.36}$$

In almost the same way as in [13], one can prove that the determinant (3.9) with the above $\tilde{F}_n(x;t)$ gives the Green's function for the DNLS-type model with $\alpha = -1$.

3.4. Model with coagulation

Since the introduction for the TASEP in [10], one of the interesting questions has been whether the determinant formula is related to some other known objects. In this section, we show that the above determinant formula can be regarded as a deformation of the Slater determinant type Green's function.

As an example, consider a totally asymmetric diffusion model for exclusive particles with the following Hamiltonian,

$$H_{\mathrm{coag}} = -\sum_j [s_j^+ s_{j+1}^- - n_j(1 - n_{j+1}) + c(s_j^+ - n_j)n_{j+1})] \tag{3.37}$$

with $0 \leqslant c \leqslant 1$. Unlike the models considered so far, there occurs a particle coagulation $11 \to 01$ with rate c. The $c = 0$ case corresponds to the TASEP. On the other hand, the $c = 1$ case is known to be solvable by the free fermion method [22]. This fact suggests that some correlation functions for the model (3.37) with $c = 1$ are written in a Slater determinant form. In fact, the conditional probability $P(x_1, \ldots, x_N; t | y_1, \ldots, y_N; 0)$ of finding N particles on lattice sites $x_1, \ldots, x_N$ at time t with initial occupation $y_1, \ldots, y_N$ at time $t = 0$ is written in the form, (3.9) and (3.10), with the choice,

$$F_n(x;t) = F_0(x;t) = \frac{t^x}{x!}\mathrm{e}^{-t} \tag{3.38}$$

for all n. This expression is indeed the Slater determinant form.

It turns out that the conditional probability $P(x_1, \ldots, x_N; t|y_1, \ldots, y_N; 0)$ for the model (3.37) with any value of c can also be written in a determinant form (3.9). The function $F_n(x; t)$ is given by

$$F_n^{(c)}(x; t) = \mathrm{e}^{-t} \sum_{l=0}^{\infty} (1-c)^l \begin{bmatrix} l-n-1 \\ l \end{bmatrix} \frac{t^{x+l}}{(x+l)!}. \tag{3.39}$$

This is a simple generalization of (3.12) and reduces to expressions (3.12) and (3.38), when $c = 0$ and $c = 1$ respectively. Hence, we can say that the determinant formula of the Green's function first introduced in [10] is regarded as a deformation of the Slater determinant for the free fermion models.

4. Bethe ansatz and two-particle diffusion for m-TASEP

In this section, we restrict out attention to the m-TASEP. We solve the eigenvalue problem by the Bethe ansatz and derive the S-matrices of the process. First we solve the two-particle problem. If we assume the time dependence as $P(x_1, x_2; t) = \mathrm{e}^{-Et} P(x_1, x_2)$ in the master equation (3.4), the resulting equation is

$$EP(x_1, x_2) = -P(x_1 - 1, x_2) - P(x_1, x_2 - 1) + 2P(x_1, x_2). \tag{4.1}$$

The boundary condition is the same as in the previous section (cf (3.6)),

$$P(x_1, x_1 + m - 1) = P(x_1, x_1 + m). \tag{4.2}$$

We assume that the wavefunction is written in the form,

$$P(x_1, x_2) = A_{12} \mathrm{e}^{\mathrm{i}p_1 x_1 + \mathrm{i}p_2 x_2} + A_{21} \mathrm{e}^{\mathrm{i}p_2 x_1 + \mathrm{i}p_1 x_2} \qquad (x_2 - x_1 \geqslant m). \tag{4.3}$$

For $m = 0$, this assumption may seem too simple. In general, we have to consider the case $x_1 = x_2$ separately. But it turns out that it is sufficient to consider the wavefunction (4.3) for the m-TASEP. In the next section we shall see that the wavefunction of the q-boson model cannot be written as a single expression (4.3) except when $q \to \infty$. From condition (4.1) for $x_2 - x_1 > m$, the energy is easily calculated as $E = \epsilon_{p_1} + \epsilon_{p_2}$, where $\epsilon_p = 1 - \mathrm{e}^{-\mathrm{i}p}$. The boundary condition (4.2) leads to the two-body S-matrix for the m-TASEP,

$$S_{12} = \frac{A_{21}}{A_{12}} = \frac{\mathrm{e}^{\mathrm{i}p_2(m-1)} - \mathrm{e}^{\mathrm{i}p_2 m}}{\mathrm{e}^{\mathrm{i}p_1(m-1)} - \mathrm{e}^{\mathrm{i}p_1 m}}. \tag{4.4}$$

Using eigenfunction (4.3), we can construct the Green's function $P(x_1, x_2; t|y_1, y_2; 0)$,

$$\begin{aligned} P(x_1, x_2; t|y_1, y_2; 0) = {} & \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \mathrm{d}p_1 \int_0^{2\pi} \mathrm{d}p_2 \, \mathrm{e}^{-(\epsilon_{p_1} + \epsilon_{p_2})t - \mathrm{i}p_1 y_1 - \mathrm{i}p_2 y_2} \\ & \times (\mathrm{e}^{\mathrm{i}p_1 x_1 + \mathrm{i}p_2 x_2} + S_{12} \mathrm{e}^{\mathrm{i}p_2 x_1 + \mathrm{i}p_1 x_2}) \end{aligned} \tag{4.5}$$

where we shift the pole in S_{12} as $p_1 \to p_1 + \mathrm{i}0$ to satisfy the initial condition.

Further, the eigenvalue problem for N particles,

$$EP(x_1, \ldots, x_N) = -\sum_{j=1}^{N} P(x_1, \ldots, x_j - 1, \ldots, x_N) + NP(x_1, \ldots, x_N) \tag{4.6}$$

can also be solved by the Bethe ansatz. For a fixed set of momenta $p_1, \ldots, p_N$, the wavefunction for the region $x_{j+1} - x_j \geqslant m$ is taken as a linear combination of the plane waves,

$$P_{p_1,\ldots,p_N}(x_1, x_2, \ldots, x_N) = \sum_{\sigma \in \mathfrak{S}_N} A_{\sigma(1)\sigma(2)\ldots\sigma(N)} \mathrm{e}^{\mathrm{i}p_{\sigma(1)}x_1 + \cdots + \mathrm{i}p_{\sigma(N)}x_N} \tag{4.7}$$

where the symbol $\mathfrak{S}_N$ denotes all permutations of N numbers $\{1, \ldots, N\}$ and σ is an element of $\mathfrak{S}_N$. If we set $A_{1\ldots N} = 1$, the coefficients $A_{\sigma(1)\ldots\sigma(N)}$ are written as

$$A_{\sigma(1)\sigma(2)\ldots\sigma(N)} = \operatorname{sgn}\sigma \prod_{j=1}^{N} (\mathrm{e}^{\mathrm{i}p_{\sigma(j)}(m-1)} - \mathrm{e}^{\mathrm{i}p_{\sigma(j)}m})^{\sigma(j)-j} \tag{4.8}$$

where $\operatorname{sgn}\sigma$ indicates the signature of the permutation σ. Thus, the wavefunction (4.7) is expressed as

$$P_{p_1,\ldots,p_N}(x_1, x_2, \ldots, x_N) = \det W \tag{4.9}$$

with the matrix elements,

$$W_{jl} = (\mathrm{e}^{\mathrm{i}p_j(m-1)} - \mathrm{e}^{\mathrm{i}p_j m})^{j-l} \mathrm{e}^{\mathrm{i}p_j x_l}. \tag{4.10}$$

Setting $p_j \to p_j + \mathrm{i}0$, we can rederive the determinant form Green's function (3.9) obtained in the previous section.

Next we study how the interaction of the m-TASEP affects the diffusion of two particles. Although the two-particle system seems trivial, it reflects the characteristic behaviours of the driven system at finite particle density. For instance, it was shown that the collective diffusion constant is enhanced for the TASEP [10]. This enhancement may be related to the superdiffusive spreading of a local inhomogeneity in a homogeneous background with finite density [2]. We show that the drift velocity and the collective diffusion constant for the m-TASEP are independent of the value of m. This fact suggests that the m-TASEP have common properties even when the particle density is large and are in the same universality class.

We introduce the expectation value $\langle n_x \rangle$ of particle number on site x at time t and define the moments of it. For the two-particle case, the first and second moments are given by

$$\begin{aligned} \langle X \rangle &= \tfrac{1}{2} \sum_x x \langle n_x \rangle = \tfrac{1}{2} \sum_{x_2 - x_1 \geqslant m} (x_1 + x_2) P(x_1, x_2; t) \\ \langle X^2 \rangle &= \tfrac{1}{2} \sum_x x^2 \langle n_x \rangle = \tfrac{1}{2} \sum_{x_2 - x_1 \geqslant m} (x_1^2 + x_2^2) P(x_1, x_2; t). \end{aligned} \tag{4.11}$$

In terms of these moments, the drift velocity v and collective diffusion constant Δ are defined as

$$\begin{aligned} v &= \frac{\mathrm{d}}{\mathrm{d}t} \langle X \rangle \\ \Delta &= \frac{\mathrm{d}}{\mathrm{d}t} (\langle X^2 \rangle - \langle X \rangle^2). \end{aligned} \tag{4.12}$$

For comparison, we note that these quantities for the free boson model (2.4) are easily calculated as

$$v = \Delta = 1 \qquad \text{(free boson case)}. \tag{4.13}$$

For the m-TASEP, the time evolution equation for $\langle n_x \rangle$ satisfies the continuity equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle n_x \rangle = \langle j_{x-1} \rangle - \langle j_x \rangle \tag{4.14}$$

with the current being

$$\langle j_x \rangle = \sum_{y(\leqslant x-m)} P(y, x; t) + \sum_{y(\geqslant x+m)} P(x, y; t) - P(x, x+m; t). \tag{4.15}$$

If we define the quantities,

$$
\begin{aligned}
P_0 &= \sum_x P(x, x+m; t) \\
P_1 &= \sum_x x P(x, x+m; t)
\end{aligned}
\tag{4.16}
$$

the drift velocity and the collective diffusion constant are rewritten as

$$
\begin{aligned}
v &= \tfrac{1}{2} \sum_x \langle j_x \rangle \\
&= 1 - \tfrac{1}{2} P_0
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
\Delta &= v - 2v\langle X \rangle + \sum_x x \langle j_x \rangle \\
&= v + 2(1-v)\langle X \rangle - P_1.
\end{aligned}
\tag{4.18}
$$

Using expression (4.5), we can calculate the asymptotic forms of P_0 and P_1,

$$
P_0 \to \frac{1}{\sqrt{\pi t}} \tag{4.19}
$$

$$
P_1 \to \sqrt{\frac{t}{\pi}} - \frac{1}{2}. \tag{4.20}
$$

Inserting these expressions into (4.17) and (4.18), we obtain

$$
v \to 1 \tag{4.21}
$$

$$
\Delta \to \frac{3}{2} - \frac{1}{\pi} \tag{4.22}
$$

as $t \to \infty$. From (4.22) and (4.13), we see that the collective diffusion constant increases by $\frac{1}{2} - 1/\pi$ as compared with the bosonic case. The enhancement does not depend on the integer m. In particular, we find that the properties of the two-particle diffusion for the phase model ($m = 0$) and the TASEP ($m = 1$) are the same asymptotically. This suggests that the two models belong to the same KPZ universality class.

5. q-boson totally asymmetric diffusion model

As noted in the introduction, the free boson model (2.4) and the phase model (2.8) are both limiting cases of the q-boson model (2.6). Here we show the integrability of the q-boson model with periodic boundary condition and study the two-particle diffusion problem on the infinite lattice.

5.1. Integrability and algebraic Bethe ansatz

Only in this section, we consider the periodic model on a finite lattice with length M. Define the L-operator for the q-boson model at lattice site j as

$$
L_j(u) = \begin{bmatrix} u^{-1} - \gamma u q^{-2N_j} & \chi B_j^\dagger \\ \chi B_j & u \end{bmatrix} \tag{5.1}
$$

where $\chi = \sqrt{1-q^{-2}}$, γ is a parameter and u is the spectral parameter. We use the standard notations such as $\overset{1}{L}_j(u) = L_j(u) \otimes 1$, $\overset{2}{L}_j(u) = 1 \otimes L_j(u)$. The L-operator satisfies the bilinear relation (Yang–Baxter relation),

$$
R^{12}(u,v)\overset{1}{L}_j(u)\overset{2}{L}_j(v) = \overset{2}{L}_j(v)\overset{1}{L}_j(u)R^{12}(u,v) \tag{5.2}
$$

for the (gauge-transformed) trigonometric R-matrix,

$$R(u,v) = \begin{bmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & q^{-1} & g(v,u) & 0 \\ 0 & g(v,u) & q & 0 \\ 0 & 0 & 0 & f(v,u) \end{bmatrix} \tag{5.3}$$

with the matrix elements defined by the functions,

$$f(v,u) = \frac{qu^2 - q^{-1}v^2}{u^2 - v^2} \qquad g(v,u) = \frac{(q-q^{-1})uv}{u^2 - v^2}. \tag{5.4}$$

The L-operator (5.1) is a generalization of the L-operator in [15], where the parameter γ is set to zero. As will be seen, we can derive the stochastic q-boson Hamiltonian (2.6) from the L-operator (5.1). In addition, it is known that the ASEP has the same R-matrix [8]. Hence we can expect that the ASEP and the q-boson model (2.6) may share common properties. Due to the locality of the L-operator, the monodromy matrix,

$$T(u) = L_M(u)\dots L_1(u) = \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix} \tag{5.5}$$

satisfies the relation,

$$R^{12}(u,v)\overset{1}{T}(u)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{T}(u)R^{12}(u,v). \tag{5.6}$$

If we introduce the transfer matrix $\tau(u)$ by

$$\tau(u) = \mathrm{Tr}\, T(u) = A(u) + D(u) \tag{5.7}$$

it follows from (5.6) that the transfer matrices with different spectral parameters commute;

$$[\tau(u), \tau(v)] = 0. \tag{5.8}$$

In addition, we find

$$-\frac{1}{\chi^2}\frac{\partial}{\partial u^2}(u^M \tau(u))|_{u=0} = -\sum_{j=1}^{M}(B_{j+1}^\dagger + \gamma B_j^\dagger)B_j + \frac{M}{\chi^2} \tag{5.9}$$

where $B_{M+1} = B_1$. Hence, the q-boson Hamiltonian (2.6) is obtained by putting $\gamma = -1$. It commutes with the transfer matrix (5.7). In this way, we have proved that the transfer matrix is a generator of mutually commuting conserved operators and the model is integrable.

The eigenvectors of the transfer matrix and hence of the Hamiltonian are of the form,

$$|\Phi(u_1,\dots,u_N)\rangle = \prod_{j=1}^{N} B(u_j)|0\rangle. \tag{5.10}$$

Here the parameters u_j in (5.10) satisfy the Bethe equations,

$$\left[\frac{a(u_j)}{d(u_j)}\right]^M = \prod_{k(\neq j)}^{N} \frac{f(u_k,u_j)}{f(u_j,u_k)} \qquad (j = 1,\dots,N) \tag{5.11}$$

with the functions $a(u)$ and $d(u)$ defined by

$$a(u) = u^{-1} - \gamma u \qquad d(u) = u. \tag{5.12}$$

The above construction of the eigenvectors is a standard procedure of the algebraic Bethe ansatz method.

5.2. Two-particle diffusion

Next we consider the two-particle diffusion problem on the infinite lattice. We are interested in how the expressions of the collective diffusion constants for the free boson model (4.13) and the phase model (4.22) are connected by the parameter q. In this section, we set $\gamma = -1$ to study the q-boson model with stochastic interpretation. Introducing the momentum variable p by

$$\mathrm{e}^{\mathrm{i}p} = \frac{d(u)}{a(u)} = \frac{1}{u^{-2}+1} \tag{5.13}$$

we write down the Bethe ansatz wavefunction for two particles from (5.10). If we normalize the wavefunction for $x_1 = x_2$ by setting

$$P(x_1, x_1) = \mathrm{e}^{\mathrm{i}(p_1+p_2)x_1} \tag{5.14}$$

the wavefunction for $x_2 - x_1 > 0$ is calculated as

$$P(x_1, x_2) = A_{12}\mathrm{e}^{\mathrm{i}p_1x_1+\mathrm{i}p_2x_2} + A_{21}\mathrm{e}^{\mathrm{i}p_2x_1+\mathrm{i}p_1x_2} \tag{5.15}$$

where

$$\begin{aligned} A_{12} &= \frac{q^{-2}\mathrm{e}^{\mathrm{i}p_1} - \mathrm{e}^{\mathrm{i}p_2} + (1-q^{-2})\mathrm{e}^{\mathrm{i}(p_1+p_2)}}{\mathrm{e}^{\mathrm{i}p_1} - \mathrm{e}^{\mathrm{i}p_2}} \\ A_{21} &= \frac{q^{-2}\mathrm{e}^{\mathrm{i}p_2} - \mathrm{e}^{\mathrm{i}p_1} + (1-q^{-2})\mathrm{e}^{\mathrm{i}(p_1+p_2)}}{\mathrm{e}^{\mathrm{i}p_2} - \mathrm{e}^{\mathrm{i}p_1}}. \end{aligned} \tag{5.16}$$

Since $A_{12} + A_{21} = 1 + q^{-2}$, we have to consider the $x_1 = x_2$ case separately except when $q \to \infty$, which corresponds to the phase model.

The two-particle Green's function $P(x_1, x_2; t|y_1, y_2; 0)$ can be constructed by integrating the eigenfunction, (5.14) and (5.15), with an appropriate coefficient. By using the contour integration, we can show

$$\begin{aligned} P(x_1, x_2; t|y_1, y_2; 0) = {} & \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \mathrm{d}p_1 \int_0^{2\pi} \mathrm{d}p_2\, \mathrm{e}^{-(\epsilon_{p_1}+\epsilon_{p_2})t - \mathrm{i}p_1y_1 - \mathrm{i}p_2y_2} \\ & \times(\mathrm{e}^{\mathrm{i}p_1x_1+\mathrm{i}p_2x_2} + S_{12}\mathrm{e}^{\mathrm{i}p_2x_1+\mathrm{i}p_1x_2}) \end{aligned} \tag{5.17}$$

with

$$S_{12} = \frac{A_{21}}{A_{12}} = \frac{q^{-2}\mathrm{e}^{\mathrm{i}p_2} - \mathrm{e}^{\mathrm{i}p_1} + (1-q^{-2})\mathrm{e}^{\mathrm{i}(p_1+p_2)}}{q^{-2}\mathrm{e}^{\mathrm{i}p_1} - \mathrm{e}^{\mathrm{i}p_2} + (1-q^{-2})\mathrm{e}^{\mathrm{i}(p_1+p_2)}} \tag{5.18}$$

for $x_2 - x_1 > 0$. To satisfy the initial condition, the pole in S_{12} is shifted as $p_j \to p_j + \mathrm{i}0$. On the other hand, for $x_1 = x_2$, we have

$$\begin{aligned} P(x, x; t|y_1, y_2; 0) = {} & \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \mathrm{d}p_1 \int_0^{2\pi} \mathrm{d}p_2\, \mathrm{e}^{-(\epsilon_{p_1}+\epsilon_{p_2})t + \mathrm{i}(p_1+p_2)x - \mathrm{i}p_1y_1 - \mathrm{i}p_2y_2} \\ & \times \frac{\mathrm{e}^{\mathrm{i}p_1} - \mathrm{e}^{\mathrm{i}p_2}}{q^{-2}\mathrm{e}^{\mathrm{i}p_1} - \mathrm{e}^{\mathrm{i}p_2} + (1-q^{-2})\mathrm{e}^{\mathrm{i}(p_1+p_2)}}. \end{aligned} \tag{5.19}$$

Compare this expression with that for the ASEP [10]. In contrast to the ASEP case, we do not have to take the bound-state contributions into account for the q-boson model. In general, there seems to be no bound states for 'totally' asymmetric diffusion models.

For the q-boson model, the current $\langle j_x \rangle$ in (4.14) is given by

$$\langle j_x \rangle = \sum_{y(>x)} P(y, x; t) + \sum_{y(<x)} P(x, y; t) + (1 + q^{-2})P(x, x; t). \tag{5.20}$$

Hence the drift velocity v and the collective diffusion constant Δ are rewritten as

$$v = 1 - \frac{1-q^{-2}}{2} P_0 \tag{5.21}$$

$$\Delta = v + 2(1-v)\langle X \rangle - (1-q^{-2}) P_1 \tag{5.22}$$

where $P_0 = \sum_x P(x,x;t)$ and $P_1 = \sum_x x P(x,x;t)$. Inserting the asymptotic forms of P_0 and P_1,

$$P_0 \to \frac{1}{(1+q^{-2})\sqrt{\pi t}} \tag{5.23}$$

$$P_1 \to \frac{1}{1+q^{-2}} \sqrt{\frac{t}{\pi}} - \frac{1-q^{-2}}{2(1+q^{-2})^2} \tag{5.24}$$

into (5.21) and (5.22), we finally find

$$\begin{aligned} v &\to 1 \\ \Delta &\to 1 + \left(\frac{1}{2} - \frac{1}{\pi}\right)\left(\frac{1-q^{-2}}{1+q^{-2}}\right)^2 \end{aligned} \tag{5.25}$$

as $t \to \infty$. This expression interpolates between the free boson case (4.13) and the phase model case, (4.21) and (4.22).

6. Concluding remarks

In this paper we have obtained some exact results for asymmetric diffusion models. First we found that the determinant formula of the N-particle Green's function is valid for a wide class of totally asymmetric diffusion models. The models with such determinant solution include the phase model, the m-TASEP, the m-drop-push model and the derivative nonlinear Schrödinger-type model with special values of the coupling constant. We have also found that such novel representation of the Green's function is regarded as a deformation of the Slater determinant-type correlation functions for the free fermion models. On the other hand, we have introduced a new model, the q-boson totally asymmetric diffusion model, which connects the free boson model and the phase model. The integrability of the model has been proved by generalizing the L-operator in [15]. The Bethe ansatz equations for the q-boson model will be analysed in future publications. It is interesting to see whether the model belongs to the KPZ universality class or not.

For the m-TASEP and the q-boson model, the two-particle diffusion problem has been studied. The collective diffusion constant for the m-TASEP has been shown to be independent of the integer m. It suggests that the phase model and the TASEP are in the same universality class. For the q-boson model, the expression for the collective diffusion constant has been obtained which interpolates the free boson model and the phase model.

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